Lecture 14

Application of covariant Mechanics and Action principle for CED

1. Recap

2. • Motion in constant Electric field (cont.) + trajectory [Previous lecture]
   • Motion in constant Magnetic field

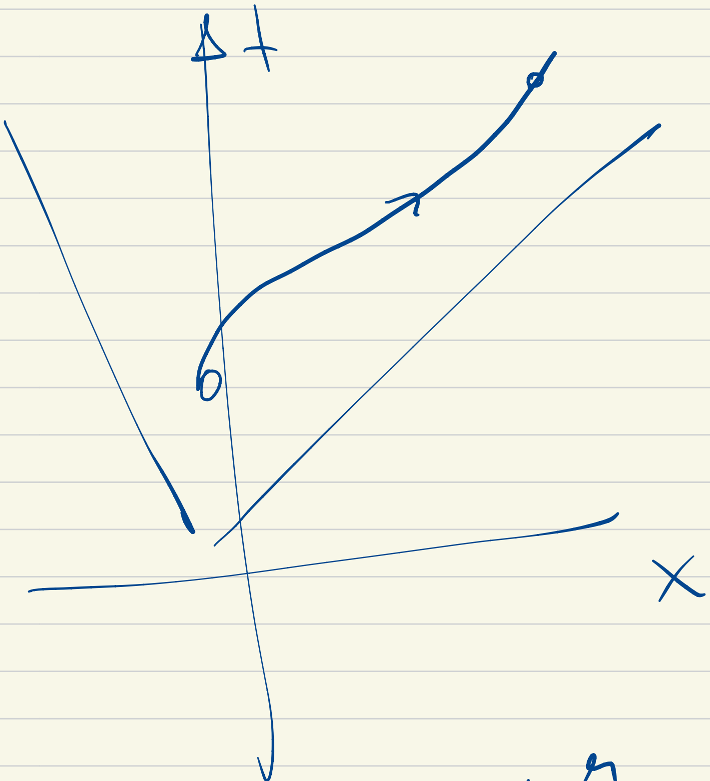
3. Radiation of a moving charge

4. Action principle for electromagnetic field.

5. Review

$$\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0$$

$$\partial_\mu F^{\mu\nu} = -\frac{1}{c\varepsilon_0} J^\nu$$



$$d\tau = \sqrt{-\dot{x}^\mu \dot{x}_\mu}\, dt$$

proper time

$$u^\mu = \frac{dx^\mu}{d\tau} = \gamma \begin{pmatrix} c \\ \vec{v} \end{pmatrix}$$

$$u^\mu u_\mu = -c^2 \quad , \quad a^\mu = \frac{du^\mu}{d\tau}$$

$$m a^\mu = \frac{q}{c} F^{\mu\nu} u_\nu \qquad \text{[Lorentz force]}$$

$$p^\mu = m u^\mu = \begin{pmatrix} \varepsilon/c \\ \vec{p} \end{pmatrix} \qquad p^\mu p_\mu = -m^2 c^2 \to \varepsilon = mc$$

Motion in constant magnetic field

$$\frac{d\vec{p}}{dt} = q(\vec{v} \times \vec{B})$$

$$\frac{d}{dt}\mathcal{E} = 0 \Rightarrow \text{const} \Rightarrow |\vec{v}| = \text{const}$$

$$\frac{d\vec{v}}{dt} = \vec{v} \times \vec{\omega}_B, \qquad \omega_B = \frac{q\vec{B}}{\gamma m} = \frac{q\vec{B}c^2}{\mathcal{E}}$$

this is called "cyclotron frequency"

$B \parallel B_z$

$$\frac{dv_x}{dt} = v_y \omega_B \qquad \frac{dv_y}{dt} = -v_x \omega_B$$

$$x(t) = -R\cos\omega_B t \qquad y(t) = R\sin\omega_B t$$

$$z(t) = v_z \cdot t$$

Radiation of a moving charge

(Relativistic generalization of Larmor formula.)

Larmor formula (lecture 5).

$$\frac{q^2}{4\pi\varepsilon_0} \dot{v}^2 = \frac{d\mathcal{E}}{dt} \quad \text{(before)}$$

$$dP^\mu = \frac{q^2}{6\pi\varepsilon_0 c^5} \left(\frac{du^\nu}{d\tau} \frac{du_\nu}{d\tau} \right) dx^\mu \rightarrow$$

$\rightarrow$ the only expression quadratic in acceleration $\left[a^\mu = \frac{du^\nu}{d\tau} \right]$

$$c^2 \frac{dP^0}{dx^0} = -\frac{q^2}{6\pi\varepsilon_0 c^3} \frac{du^\nu}{d\tau} \frac{du_\nu}{d\tau} =$$

$$= \frac{q^2}{6\pi\varepsilon_0 c^3}\gamma^6\left[|\dot{\vec{\beta}}|^2 - |\beta\times\dot{\vec{\beta}}|^2\right] \qquad (*)$$

Before that, after a very tedious work with Liénard-Wiechert Potentials we derived

$$\frac{d\mathcal{E}}{d\Omega\, dt} = \frac{q^2}{16\pi^2\varepsilon_0 c}\frac{\left|\vec{n}\times((\vec{n}-\vec{\beta})\times\dot{\vec{\beta}})\right|^2}{(1-\vec{n}\cdot\vec{\beta})^6}$$

if we integrate it over the angles $d\Omega$ we will recover $(*)$.

Action principle for electromagnetic field and for a charged particle.

$$S = S_f + S_m + S_{mf}$$

$$S_m = S_{particle} = -\int d\lambda \, mc \sqrt{-\dot{x}^\mu \dot{x}_\mu}$$

$\lambda \to f(\lambda)$ gauge inv.

$$S_{mf} = -\frac{q}{c} \int d\lambda \, A_\mu(x(\lambda)) \, \dot{x}^\mu = -\frac{1}{c} \int A^\mu j_\mu$$

$$\boxed{\begin{aligned} S_f &= -\frac{\varepsilon_0}{4} \int d^4x \, F^{\mu\nu} F_{\mu\nu} \\ F^{\mu\nu} F_{\mu\nu} &\simeq E^2 - H^2 \end{aligned}}$$

- Derive Maxwell equations
- Derive Lorentz force:

integrated by parts

$$-\frac{q}{c}\int d\lambda\, \partial_\nu A_\mu \delta x^\nu \cdot \cancel{\dot{x}}^\mu + \frac{q}{c}\int d\lambda\, \dot{x}^\nu \cdot$$

$$\cdot\ \partial_\nu A_\mu \delta x^\mu \;\underbrace{-\int d\lambda\, \frac{1}{2}\frac{\cdot 2\cancel{\dot{x}}_\mu \cdot \delta\dot{x}^\mu}{\sqrt{-\dot{x}^\nu \dot{x}_\nu}} \cdot m^2 c}$$

last term also need to integrate by parts:

$$\int d\lambda\, \frac{1}{2}\frac{\cdot 2\cancel{\dot{x}}_\mu \cdot \delta\dot{x}^\mu}{\sqrt{-\dot{x}^\nu \dot{x}_\nu}} \cdot m^2 c^2 = -mc^2 \int d\lambda\, \frac{d}{d\lambda}\left(\frac{\dot{x}_\mu}{\sqrt{-\dot{x}^\mu \dot{x}_\mu}}\right)\delta\cancel{\dot{x}}^\mu$$

$$cm^2 \frac{d}{d\lambda}\left[\frac{\dot{x}_\mu}{\sqrt{-\dot{x}^\nu \dot{x}_\nu}}\right] = \left(\partial_\mu A_\nu \dot{x}^\nu - \partial_\nu A_\mu \dot{x}^\nu\right)\frac{q}{c} =$$

$$= \frac{q}{c} F_{\mu\nu} \dot{x}^\nu$$

choose $\lambda = t$ and multiply both parts by $\frac{1}{\gamma} = \frac{c}{\sqrt{-\dot{x}^\mu \dot{x}_\mu}}$

$$ds = \sqrt{1-\vec{\beta}^2}\, dt \quad \Rightarrow \quad \frac{1}{\gamma}\frac{d}{dt} = \frac{1}{ds}$$

- When we vary the action with respect to A_μ we get:

$$\int d^4x \left(\varepsilon_0 \underbrace{\partial_\mu F^{\mu\nu}}_{\text{four terms!}} - \frac{1}{c} J^\nu\right)\delta A_\nu$$

Maxwell equation!

since we work with A_μ

$\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma}$ is automatic.

Review

Maxwell equations:

$$\text{I} \quad \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\text{II} \quad \vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\text{III} \quad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\text{IV} \quad \vec{\nabla}\cdot\vec{B} = 0$$

III and IV guarantee that we can express $\vec{E}$ and $\vec{B}$ through the potentials:

$$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

For given $\vec{E}$ and $\vec{B}$ there is ambiguity in $\vec{A}$ and φ called gauge invariance:

$$\begin{cases} \vec{A} \to \vec{A} + \vec{\nabla}\alpha \\ \varphi \to \varphi - \partial_t \alpha \end{cases}$$ leaves $\vec{E}$ and $\vec{B}$ invariant

For example, adding arbitrary constants to the potentials $[\,\alpha(\vec{x},t) = C_i x_i + C_0 t\,]$ does not have any physical consequences.

We can fix the gauge by using Lorentz gauge condition:

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = 0$$

Then I and II Maxwell eqns become:

$$\Box\varphi = \frac{\rho}{\varepsilon_0}$$
$$\Box\vec{A} = \mu_0\vec{j}$$

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta$$

[III and IV are satisfied automatically]

- It is often easier to find φ and $\vec{A}$ in a given problem and then derive $\vec{E}$ and $\vec{B}$ using

$$\begin{cases} \vec{E} = -\vec{\nabla}\varphi - \dfrac{\partial \vec{A}}{\partial t} \\ \vec{B} = \vec{\nabla} \times \vec{A} \end{cases}$$

- We considered the following types of problems:

	Static Δ	Dynamical $\Box$
homogeneous $\rho, \vec{j} = 0$	trivial	Plane waves
inhomogeneous $\rho, \vec{j} \neq 0$	• Boundary problems • Multipole exp.	• Radiation of EM waves • Multipole exp.

Boundary problems

Electric

$$\Delta \varphi = -\frac{\rho}{\varepsilon_0}$$

Magnetic

$$\Delta \vec{A} = -\vec{J}\mu_0$$

$$\vec{\nabla}\cdot\vec{A} = \vec{\nabla}\cdot\vec{J} = 0$$

Boundary problems

Image Method

("Simple" boundary and sources)

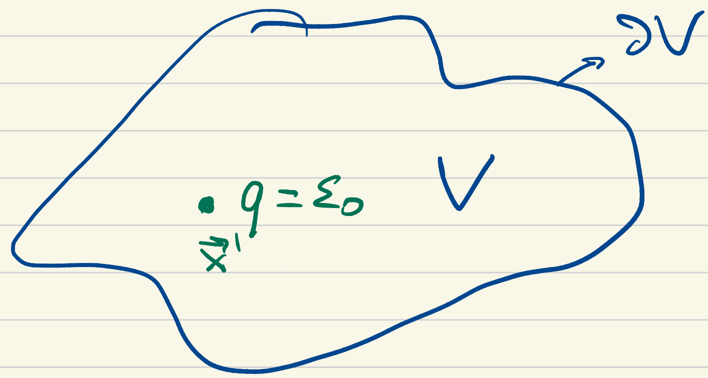
Greens function

("Generic" boundary and sources)

Greens function:

$$\Delta G(\vec{x}, \vec{x}') = -\delta^3(\vec{x}-\vec{x}') \qquad (*)$$

+ Boundary conditions



(*) is satisfied only inside V

$$G = \frac{1}{4\pi|\vec{x}-\vec{x}'|} + F, \quad \Delta F = 0$$

inside V

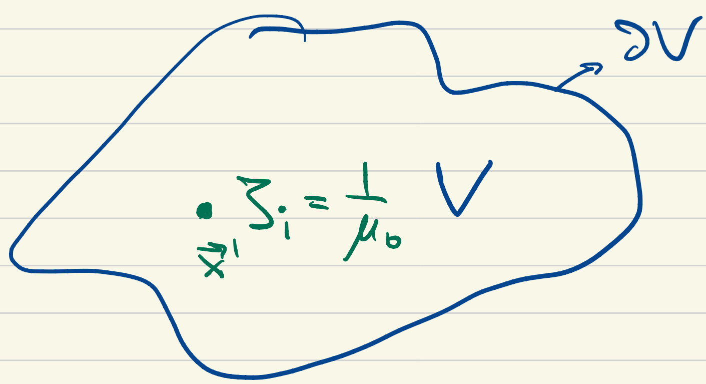
Physical meaning of G:

Potential of a charge $q=\varepsilon_0$ at a point $\vec{x}'$, plus of the charges outside of V or on the boundary, that together produce the needed boundary conditions.

Works equivalently for magnetic case:

Vector Potential of a ~~charge~~ current $\cancel{q=\varepsilon_0}$ $J_i = \frac{1}{\mu_0}$ at a point $\vec{x}'$, plus of the ~~charges~~ currents outside of V or on the boundary, that together produce the needed boundary conditions.

$$\Delta A_i = -J_i \mu_0$$



Commonly used B.C. are Neumann or Dirichlet:

$$G^D\Big|_{\partial V} = 0 \qquad \varphi\Big|_{\partial V} = \varphi_0 :$$

$$\varphi(\vec{x}) = \frac{1}{\varepsilon_0}\int_V d^3x'\, G^D(\vec{x},\vec{x}')\,\rho(\vec{x}') -$$

$\vec{x} \in V$

$$- \oint_{\partial V} d^2\vec{S}\cdot\vec{\nabla}\, G^D(\vec{x},\vec{x}')\,\varphi_0(\vec{x}')$$

$\vec{x}' \in \partial V$

Neumann:

$$\frac{\partial}{\partial n'} G^N(\vec{x},\vec{x}')\Big|_{\partial V} = 0 \qquad \frac{\partial}{\partial n}\varphi\Big|_{\partial V} = g :$$

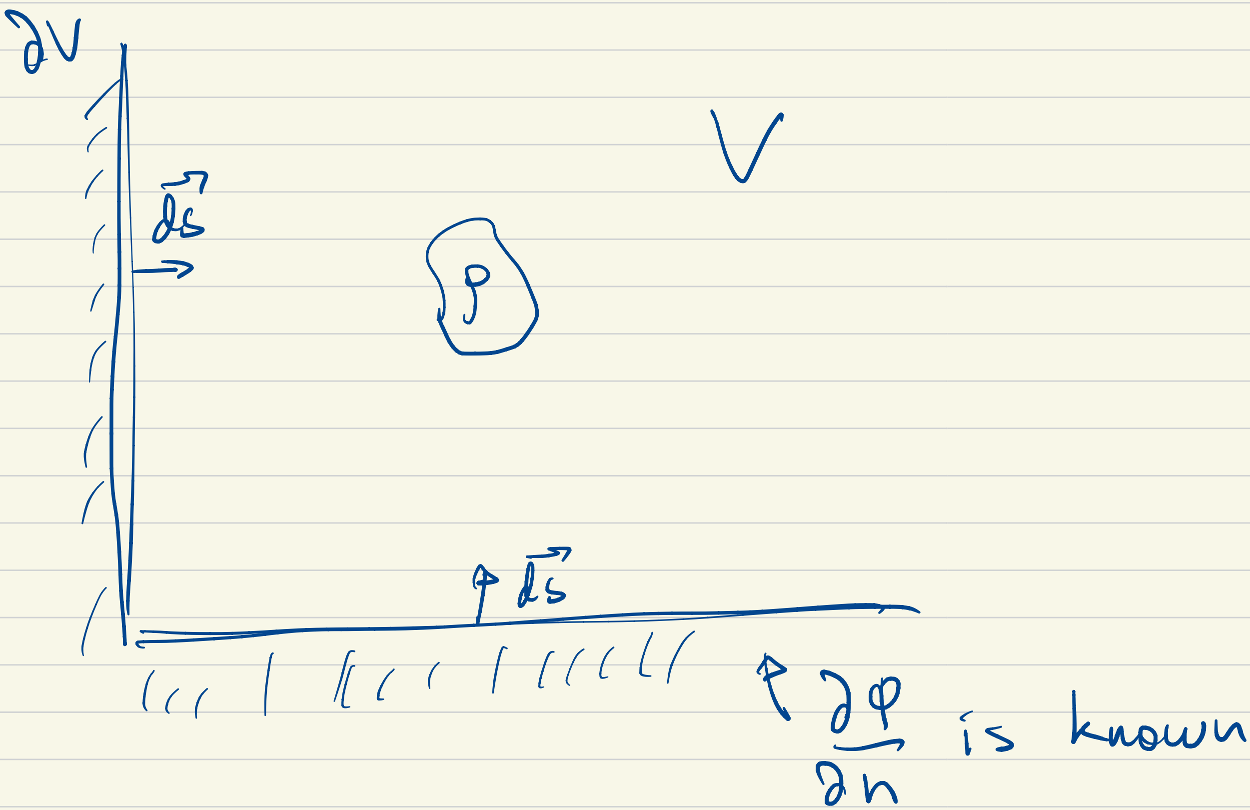
$$\varphi(\vec{x}) = \frac{1}{\varepsilon_0}\int_V d^3x'\, G^N(\vec{x},\vec{x}')\,\rho(\vec{x}') +$$

$x \in V$

$$+ \oint_{\partial V} d^2x'\, g(x')\, G^N(\vec{x},\vec{x}') \quad *$$

$\vec{x}' \in \partial V$

" Here we assumed an open Boundary"

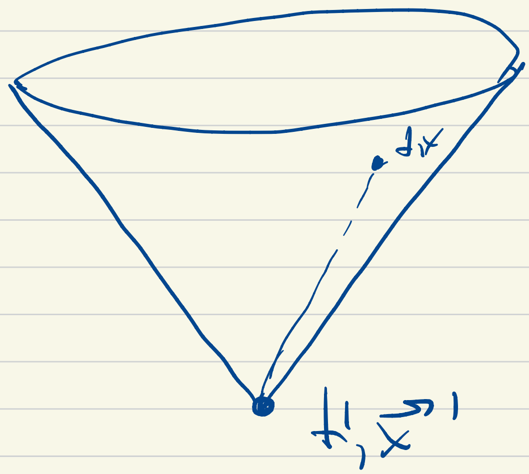


- Radiation of EM waves:

$$\Box G_{ret}(\vec{x}, t, \vec{x}', t') = \delta(t-t')\,\delta^3(\vec{x}-\vec{x}')$$

$$G_{ret} = \frac{\delta\left(t-t'+\frac{|\vec{x}-\vec{x}'|}{c}\right)}{4\pi\,|\vec{x}-\vec{x}'|}$$

localized on the lightcone:



$(t-t')\cdot c = |\vec{x}-\vec{x}\,'|$

$t > t'$

Physical meaning: Electric (Magnetic) field produced by a point-like instantaneous charge (current)

$\Box \varphi = \frac{\rho}{\varepsilon_0}$ $\Box \vec{A} = \mu_0 \vec{J}$

$$\varphi(\vec{x},t) = \frac{1}{\varepsilon_0}\int \rho(\vec{x}\,',t')\cdot G_{ret}(\vec{x},t,\vec{x}\,',t')$$

In medium:

$$\vec{\nabla} \cdot \vec{D} = \rho$$

$$\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

polarization

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \equiv \varepsilon \vec{E}$$

$$\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M} \equiv \frac{\vec{B}}{\mu}$$

magnetization

+ Matching conditions

- We found the multipole expansion in the static case:

$$\varphi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \sum_{n=0}^{\infty} \frac{1}{n!} Q_{i_1 \ldots i_n} \frac{x_{i_1} \ldots x_{i_n}}{|\vec{x}|^{2n+1}}$$

summed over

$i_k = 1 \ldots 3$

$Q_{i_1 \ldots i_n}$ = 2-n-pole moment tensor of the charge density:

$$Q_{i_1 \ldots i_n} = \int d^3x' \rho(x') T_{i_1 \ldots i_n}(x')$$

$$T_{i_1 \ldots i_n} = (2n-1)!! \, x_{i_1} x_{i_2} \ldots x_{i_n} - A_{i_1 \ldots i_n}$$

$A_{i_1 \ldots i_n}$ contains δ_{ij}'s so that to make $T_{i_1 \ldots i_n}$ traceless with respect to all indices.